① Математическое ожидание аддитивно:

$$E(M+N) = \sum_{m,n\geq 0} P_{MN}(m,n)\cdot(m+n) = \sum_{m\geq 0} m \sum_{n\geq 0} P_{MN}(m,n) + \sum_{n\geq 0} n \sum_{m\geq 0} P_{MN}(m,n) =$$

$$= \sum_{m\geq 0} m\, P_M(m) + \sum_{n\geq 0} n\, P_N(n) = E(M) + E(N)$$

def $M \perp N$: $P(M=m|N=n) = P_M(\ldots$

$M\perp N \Leftrightarrow P_{MN}(m,n) = P_M(m) P\ldots$

② Если $M\perp N$, то $E(MN) = E(M)E(N)$

$$E(MN) = \sum_{m,n\geq 0} mn\, P_{MN}(m,n) \overset{M\perp N}{=} \sum_{m\geq 0}\sum_{n\geq 0} mn\, P_M(m) P_N(N) = \Big(\sum_{n\geq 0} n\, P_N(n)\Big) \sum_{m\geq 0} m\, P_M(m) =$$

$$= E(M)E(N)$$

③ Если $M\perp N$, то $D(M+N) = D(M)+D(N)$

$$D(M+N) = E(M+N)^2 - (E(M+N))^2 = E(M^2 + 2MN + N^2) - ((EM)^2 + 2EMEN + (E\ldots$$

$$= (EM^2 - (EM)^2) + (EN^2 - (EN)^2) + 2(E(MN) - EM\,EN) \overset{M\perp N}{=} D(M) + D(N)$$

④ Распределение Бернулли

$$p(x) = \begin{cases} p, & x=1 \\ 1-p, & x=0 \end{cases} \qquad G_X(z) = \sum_{x=\{0,1\}} P_X(x)\cdot z^x = p\cdot z^1 + (1-p) z^0 =$$

$$= pz + 1 - p \quad \text{— производящая функция}$$

$$\Psi_X(s) = \sum_{x=\{0,1\}} P_X(x)\cdot e^{sx} = p\cdot e^s + 1 - p \quad \text{— производящая функция моментов}$$

⑤ Биномиальное распределение

а) Сумма ~~n~~ величин Бернулли $\sum_{i=1}^n X_i = N$

$$G_N(z) = (G_X(z))^n = (pz + (1-p))^n = \sum_{k=0}^n C_n^k p^k (1-p)^{n-k} z^k$$

$C_n^k p^k (1-p)^{n-k} = p(n=k)$

б) $$EN = G_N'(z)\Big|_{z=1} = \left[(pz+(1-p))^n\right]'\Big|_{z=1} = np\,(pz+(1-p))^{n-1}\Big|_{z=1} =$$

$$= n\cdot p\cdot(p\cdot 1 + 1 - p)^{n-1} = np$$

$$\Psi_N = (p\cdot e^s + 1 - p)^n$$

$$EN^2 = \frac{d^2\Psi_N}{ds^2}\Big|_{s=0} = \left[npe^s(pe^s + (1-p))^{n-1}\right]'\Big|_{s=0} = np\,e^s(n-1)pe^s(pe^s+(1-p))^{n-2} +$$

$$+ npe^s(pe^s + (1-p))^{n-1} = n(n-1)p^2 + np$$

$DN = EN^2 - (EN)^2 = n(n-1)p^2 + np - (np)^2 = \cancel{n^2p^2} - np^2 + np - \cancel{n^2p^2} =$
$= np(1-p)$

⑥ Распределение Пуассона

$n \to \infty,\ p \to 0,\ np \to \lambda > 0$

ф-я моментов: $(pe^s + (1-p))^n = \sum_{k=0}^{n} C_n^k p^k e^{sk} (1-p)^{n-k} = \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} \left(\frac{\lambda}{n}\right)^k e^{sk} \cdot \left(1-\frac{\lambda}{n}\right)^{n-k} =$

$= \sum_{k=0}^{n} \frac{n!}{(n-k)!\,n^k} \frac{\lambda^k}{k!} e^{sk} \left[\left(1-\frac{\lambda}{n}\right)^{n/\lambda}\right]^{\lambda} \cdot \left(1-\frac{\lambda}{n}\right)^{-k} = \sum_{k=0}^{n} \frac{(n-k+1)(n-k+2)\ldots n}{n^k} \frac{\lambda^k}{k!} e^{sk} \ldots$

$\overset{n\to\infty}{=} \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} e^{sk} \cdot e^{-\lambda} = e^{-\lambda} \sum_{k=0}^{\infty} \frac{(\lambda e^s)^k}{k!} = e^{-\lambda} e^{\lambda e^s} = e^{\lambda(e^s - 1)}$

$EN = (\psi_N)' \big|_{s=0} = \left(e^{\lambda(e^s-1)}\right)' \big|_{s=0} = \lambda e^s \cdot e^{\lambda(e^s-1)} \big|_{s=0} = \lambda e^0 \cdot e^{\lambda(e^0-1)} = \lambda$

$EN^2 = (\psi_N)'' \big|_{s=0} = \left(\lambda e^s e^{\lambda(e^s-1)}\right)' \big|_{s=0} = \lambda e^s \cdot e^{\lambda(e^s-1)} + \lambda e^s \lambda e^s e^{\lambda(e^s-1)}$
$= \lambda + \lambda^2$

$(EN)^2 = \lambda^2 \Rightarrow DN = EN^2 - (EN)^2 = \lambda + \lambda^2 - \lambda^2 = \lambda$

$P_N(n) = \frac{\lambda^k}{k!} e^{-\lambda}$ – из функции моментов

⑦ Геометрическое распределение

Сколько нужно сделать испытаний Бернулли до первого успеха?

$P_N(n) = (1-p)^{n-1} p$ – $(n-1)$ неудача и один успех в конце

$\psi_N = \sum_{n=1}^{\infty} p(1-p)^{n-1} e^{sn} = \sum_{y=0}^{\infty} p e^s (1-p)^y e^{ys} = p \cdot e^s \sum_{n=0}^{\infty} (1-p)^n e^{ns} = pe^s \frac{1}{1-(1-p)e^s}$

$y = n-1$

$\psi_N' \big|_{s=0} = pe^s \cdot \frac{1}{1-(1-p)e^s} + pe^s \cdot \left(-\frac{1}{(1-(1-p)e^s)^2} \cdot (0 - (1-p)e^s)\right) =$

$= pe^0 \cdot \frac{1}{1-(1-p)} + p \cdot e^0 \cdot \frac{(1-p)e^0}{(1-(1-p)e^0)^2} = \frac{p}{p} + \frac{p(1-p)}{p^2} = 1 + \frac{1-p}{p} = \frac{p+1-p}{p} = \frac{1}{p}$

$$N''\big|_{s=0} = \left[\frac{pe^s(1-(1-p)e^s)}{(1-(1-p)e^s)^2} + \frac{pe^{2s}(1-p)}{(1-(1-p)e^s)^2}\right]' = \left[\frac{pe^s - \cancel{p(1-p)e^{2s}} + \cancel{p(1-p)e^{2s}}}{(1-(1-p)e^s)^2}\right]'$$

$$= \frac{pe^s}{(1-(1-p)e^s)^2} - \frac{2pe^s}{(1-(1-p)e^s)^{+3}} \cdot (-(1-p)e^s) = \frac{pe^s}{(1-(1-p)e^s)^2} + \frac{2p(1-p)\ldots}{(1-(1-p)\ldots}$$

$$= \frac{p}{p^2} + \frac{2p(1-p)}{p^3} = \frac{p}{p^2} + \frac{2(1-p)}{p^2} = \frac{p+2-2p}{p^2} = \frac{2-p}{p^2}$$

$$DN = \frac{2-p}{p^2} - \frac{1}{p^2} = \boxed{\frac{1-p}{p^2}}$$

8 Обратное биномиальное распределение

Сколько нужно сделать испытаний Бернулли чтобы было ровно k успехов (до k-о

$$P_N = C_{n-1}^{k-1}\, p^k (1-p)^{n-k}$$

Это ни что иное, как сумма k геометрических величин

$$\Psi_N = \left[\frac{pe^s}{1-(1-p)e^s}\right]^k$$

С другой стороны, $\Psi_N = \sum_{n=k}^{\infty} C_{n-1}^{k-1}\, p^k (1-p)^{n-k} e^{sn} = \sum_{y=0}^{\infty} C_{y+k-1}^{k-1}\, p^k (1-p)^{y+k-k} \cdot e^{s(y+k)} =$

$y = n-k$

$n = y+k$

$$= \sum_{n=0}^{\infty} C_{n+k-1}^{k-1}\, p^k (1-p)^n e^{sn} e^{sk} = (pe^s)^k \cdot \sum_{n=0}^{\infty} C_{n+k-1}^{k-1} (1-p)^n e^{sn} =$$

$$= (pe^s)^k \sum_{n=0}^{\infty} (e^s(1-p))^n C_{n+k-1}^{k-1} = \left[\frac{pe^s}{1-(1-p)e^s}\right]^k \cdot \sum_{n=0}^{\infty} C_{n+k-1}^{k-1} (e^s(1-p))^n (1-(1-\ldots$$

Лучше так: $\sum_{n=k}^{\infty} C_{n-k}^{k-1}\, p^k (1-p)^{n-k} e^{sn} = \frac{p^k}{(1-(1-p)e^s)^k} \cdot \sum_{n=k}^{\infty} C_{n-k}^{k-1} ((1-p)e^s)^{n-k} e^{\ldots} \cdot (1-(1-p)e^s)^k$

$$= \frac{p^k e^{sk}}{(1-(1-p)e^s)^k} \sum_{n=k}^{\infty} C_{n-k}^{k-1} ((1-p)e^s)^{n-k} \cdot (1-(1-p)e^s)^k$$

$\underbrace{q}_{\text{некоторая вероятность}} = 1-(1-p)e^s \Rightarrow \sum_{n=k}^{\infty} C_{n-k}^{k-1}\, q^k (1-q)^{n-k} \Rightarrow$ тоже 1.

$$EN = \sum_{n=1}^{\infty} p(1-p)^{n-1} \cdot n = p\sum_{n=1}^{\infty} n(1-p)^{n-1} = p(-1)\, d\left(\sum_{n=1}^{\infty}(1-p)^n\right)/dp = p(-1)\left(\sum_{n=0}^{\infty}(1-p)^n - (1-p)\right.$$

$$= p(-1)\left(\frac{1}{1-(1-p)} - 1\right)' = p(-1)\left(\frac{1}{p} - 1\right)' = p\cdot(-1)\left(-\frac{1}{p^2}\right) = \frac{1}{p}$$

$$EN^2 = \sum_{n=1}^{\infty} n^2(1-p)^{n-1} \cdot p = \sum_{n=1}^{\infty}(n + n(n-1))(1-p)\cdot p\,(1-p)^{n-2} = p\cdot(1-p)\sum_{n=1}^{\infty} n(n-1)(1-p)^{n-2} +$$

$$+ p\sum_{n=1}^{\infty} n(1-p)^{n-1} = p\cdot(1-p)(-1)^2\left(\sum_{n=1}^{\infty}(1-p)^n\right)'' + \frac{1}{p} = p(1-p)\left(\frac{1}{1-(1-p)} - (1-p)^0\right)'' + \frac{1}{p} =$$

$$= p\cdot(1-p)\left(\frac{1}{p} - 1\right)'' + \frac{1}{p} = p(1-p)\cdot\left(-\frac{1}{p^2}\right)' + \frac{1}{p} = p(1-p)\frac{2}{p^3} + \frac{1}{p} = \frac{2(1-p)}{p^2} + \frac{p}{p^2} = \frac{2-p}{p^2}$$

$$DN = EN^2 - (EN)^2 = \frac{2-p}{p^2} - \frac{1}{p^2} = \frac{1-p}{p^2}.$$